International Journal of Theoretical Physics, Vol. 23, No. 12, 1984

Modalities and Quantum Mechanics

F. J. Burghardt

Institut für Theoretische Physik, Universität zu Köln, Köln, West Germany

Received July 3, 1981

Three approaches concerning the usage of modalities in the language of quantum mechanics were considered; Mittelstaedt and I built up a dialog semantics for modalities on a metalinguistic level, and a calculus of quantum modal logic is known that is complete and sound with respect to this dialogic semantics. Van Fraassen replaced the usual interpretation of quantum mechanics (with the projection postulate) by his "modal interpretation" based on a modal object language. Dalla Chiara translated a nonmodal object language for quantum mechanics and the appropriate quantum logic into a modal language. Specifically we are interested in the similarities and the differences of these three approaches.

1. INTRODUCTION

The history of modal logic began with the work of C. I. Lewis in 1912, who introduced a strict implication instead of the material implication in the logical system of Whitehead–Russel's *Principia Mathematica* or, equivalently, enriched the syntax by a new symbol for a modality, e.g., for the necessity from which other modalities can be derived. This approach was favored by most of the succeeding logicians, e.g., S. Kripke, who introduced a Leibnizian *possible world semantics* for modalities in which a proposition A is necessary iff A is true in every world which is possible relative to A. The results of this *axiomatic modal logic* are summarized in Hughes–Cresswell (1968).

Refusing a statement of Wittgenstein's *Tractatus Logico-philosophicus*, viz. that there is no metalanguage for the one and only object language, R. Carnap (1934) criticized also Lewis' extension of the object language and contended the possibility of expressing the modalities *necessary* and *possible* within the metalanguage without using a new symbol for these modalities. P. Lorenzen (1954) continued the idea of Carnap and defined a necessity relative to a given knowledge W: A proposition A is *necessary relative to W* iff W implies A. If this implication is purely logical the necessity of A is called *logical*, otherwise, e.g., using physical laws, A is called *real necessary* relative to W. In the sense of Carnap and Lorenzen *necessity* is not a property of an object proposition but of a metaproposition that is called *necessary* iff it is true with respect to a metasemantics for all knowledges W.

Further historical notes can be found in Burghardt (1979).

So, if one wishes to use modalities in the language of quantum mechanics there are two ways to do so:

(1) The object language is nonmodal but a relative necessity with respect to the knowledge about the physical system in question, e.g., with respect to its pure state, can be formulated. Given a semantics for the metalanguage necessary metapropositions can be investigated. This way is favored by Mittelstaedt and myself and will be considered in Section 2.

(2) The syntax of the object language concerning propositions A about a physical system includes a symbol for a modality, e.g., L for "necessary," and a semantics must be given for the proposition LA, "necessary A." This was done by van Fraassen, whose approach will be discussed in Section 3.

At last in Section 4 we summarize the approach of Dalla Chiara, who has a nonmodal syntax for the language about a quantum mechanical system. This language is translated into an axiomatic modal language by which Dalla Chiara's calculus of quantum logic becomes a modal calculus.

So we have distinguished the approaches of Mittelstaedt, van Fraassen, and Dalla Chiara from a formal point of view, viz., by the different levels of languages. But the aims of the three authors are different too:

(1) Mittelstaedt tries to construct a language of quantum physics by investigating the logical foundation including modal logic by which he has also a connection with probability theory.

(2) Refusing the projection and the ignorance postulate of quantum mechanics van Fraassen builds up a modal interpretation in order to get a language whereby he is able to interpret quantum mechanics, e.g., the measurement problem, "as if projection- and ignorance-postulates were true."

(3) Dalla Chiara does not use modalities within the language about a quantum mechanical system. As is well known this language and the appropriate calculus are nonclassical. Dalla Chiara translates this quantum logic into a modal logic whose calculus has a "classical" form, in order to show that "from a logical point of view quantum logic is not really essential to the logical development of quantum mechanics." So modalities only serve for looking at quantum logic from another point of view.

Throughout this paper I use three different symbols for the *necessity* (and the *possibility*): $\vartriangle$ (and $\triangledown$) in the approach of Mittelstaedt, $\Box$ (and $\Diamond$)

in the approach of van Fraassen, and L (and M) in the approach of Dalla Chiara.

2. MODAL QUANTUM LOGIC WITHIN THE METALANGUAGE (MITTELSTAEDT)

2.1. Object Language and Relative Necessity. In the object language of Mittelstaedt (1978) propositions are considered in a *material dialog* D_m in which the *proponent* P states a proposition and must defend it against the attacks of the *opponent* O. The development of a dialog is laid down by some rules, e.g., Table I. If the proponent stating the proposition A has a strategy of success for A within the material dialog irrespective of the arguments of O the proposition A is said to be *materially true* and we write $\underset{D_m}{\vdash} A$.

In the *formal dialog* D_f attacks on elementary propositions are not allowed and commensurability propositions $k(A, B)$ are replaced by the propositions $A \rightarrow (B \rightarrow A)$. If the proponent has a strategy of success for A in the formal dialog, this proposition is *formally true* which is expressed by the symbol $\underset{D_f}{\vdash} A$.

All formally true propositions can be deduced in the calculus Q_{eff} of *effective quantum logic* that is complete and sound with respect to the dialogic semantics. Extending Q_{eff} by the *tertium non datur* $A \vee \neg A$ (principle of excluded middle) one gets the calculus Q of *full quantum logic* the Lindenbaum–Tarski algebra of which is an orthocomplemented quasimodular lattice.

If a figure $A \leqslant B$ is deducible in Q_{eff} the appropriate implication $A \rightarrow B$ is formally true:

$$\underset{Q_{\text{eff}}}{\vdash} A \leqslant B \cap \underset{D_f}{\vdash} A \rightarrow B$$

The proposition A is formally true iff the implication $V \rightarrow A$ is formally true; V is the *object verum* not attackable in a dialog. We have the following:

$$\underset{Q_{\text{eff}}}{\vdash} V \leqslant A \cap \underset{D_f}{\vdash} A$$

If a proposition W representing our knowledge of the preparation of a physical system is presupposed in a dialog as being true and if furthermore the proponent has a strategy of success for a proposition A using the

TABLE I

Proposition	Symbol	Possibilities of attack	Possibilities of defense
Elementary proposition	a	$a?$	$a!$
Conjunction	$A \wedge B$	$1?, 2?$ $k(A, B)?$	$A, B,$ $k(A, B)!$
Disjunction	$A \vee B$	$?$	$A, B, \bar{k}(A, B)!$
Material implication	$A \rightarrow B$	$A, k(A, B)?$	$B, k(A, B)!$
Negation	$\neg A$	A	

knowledge W we write

$$W \underset{D_f}{\vdash} A \quad \text{or equivalently} \quad \underset{D_f}{\vdash} W \rightarrow A$$

In a material dialog we use the symbol $\Delta_W A$ instead of $W \underset{D_m}{\vdash} A$; in this case A is called *necessary relative to W*.

It is possible to replace the material dialog D_m by the formal dialog D_f adding to the latter the *factual implications* $A_j \rightarrow B_j$ that represent the contingent laws $\Gamma_i^{(q)}$ of quantum mechanics [see Mittelstaedt (1979a)]. Extending the calculus Q_{eff} by the corresponding *factual beginnings* $V \leqslant A_j \rightarrow B_j$ we get the calculus $Q_{\text{eff}}^{(f)}$. The relations between $Q_{\text{eff}}^{(f)}$ and some other well-known logical calculi can be found in Burghardt (1980).

Concerning the relative necessity of a proposition A we have

$$\Delta_W A : \bigcap W \underset{D_m}{\vdash} A \bigcap \underset{D_m}{\vdash} W \rightarrow A$$

$$\bigcap \forall_j \left(A_j \rightarrow B_j \right) \underset{D_f}{\vdash} W \rightarrow A$$

$$\bigcap \underset{Q_{\text{eff}}^{(f)}}{\vdash} W \leqslant A$$

2.2. Metalanguage and Necessary Metapropositions. As mentioned in the introduction Mittelstaedt defines the necessity within the metalanguage.

The *elementary metapropositions*

$$\text{“}A \text{ is materially true”}: \underset{D_m}{\vdash} A \quad \left(\bigcap \underset{Q_{\text{eff}}^{(f)}}{\vdash} V \leqslant A \right)$$

and

$$\text{“}A \text{ is formally true”}: \underset{D_f}{\vdash} A \quad \left(\cap \underset{Q_{\text{eff}}}{\vdash} V \leqslant A\right)$$

are *proof definite*, i.e., if a deduction of the figure $V \leqslant A$ is given that should prove the truth of the elementary metaproposition it is possible to check this deduction by the rules of the calculus Q_{eff}.

As in the object language connected metapropositions are *dialog definite*, i.e., there is a metadialog laid down by some rules in which the metaproposition can be stated by the proponent as an initial argument. One of the rules is given in Table II. The defense $\bar{a}!$ is a deduction of the figure $V \leqslant A$ within the calculus Q_{eff} or $Q_{\text{eff}}^{(f)}$ resp. In the following we use the symbol

$$V \lesssim A \quad \text{instead of} \quad \underset{Q_{\text{eff}}^{(f)}}{\vdash} V \leqslant A$$

[Remark: In Burghardt (1979, 1980) I used the symbol $V \lesssim A$ for $\underset{Q_{\text{eff}}}{\vdash} V \leqslant A$, but this is only a special case of $\underset{Q_{\text{eff}}^{(f)}}{\vdash} V \leqslant A$.] Now we consider metapropositions $\bar{A}^j(W \lesssim A_i, V \lesssim B_k)$ whose elementary metapropositions are $W \lesssim A_i$ and $V \lesssim B_k$.

(2.1) *Definition.* $\bar{A}^j(W \lesssim A_i, V \lesssim B_k)$ is *necessary* iff $\bar{A}^j$ is materially true for all W, i.e., the proponent has a strategy of success for $\bar{A}^j$ within the material metadialog for all knowledges W.

In Burghardt (1980) I proved that there is a calculus $\tilde{M}_{\text{eff}}$ which is complete and sound with respect to the metadialog semantics restricted to necessary propositions, i.e., a metaproposition $\bar{A}^j$ is necessary iff the figure $\underline{V} \leqq \bar{A}^j$ is deducible in $\tilde{M}_{\text{eff}}$. ($\underline{V}$ is the *meta verum*, a metaproposition not attackable in a metadialog.) This calculus consists of three parts:

TABLE II

Metaproposition	Symbol	Possibilities of attack	Possibilities of defense
Elementary metaproposition	$\bar{a}$	$\bar{a}?$	$\bar{a}!$
Metaconjunction	$\bar{A} \barwedge \bar{B}$	1?, 2?	$\bar{A}, \bar{B}$
Metadisjunction	$\bar{A} \veebar \bar{B}$	?	$\bar{A}$ or $\bar{B}$
Material meta-metaimplication	$\bar{A} \Rightarrow \bar{B}$	$\bar{A}$	B
Metanegation	$\urcorner\bar{A}$	$\bar{A}$	

(a) The intuitionistic calculus of formal quantum meta logic yet mentioned by Mittelstaedt (1979a) covers all metapropositions that are true with respect to a formal metadialog in which the elementary metapropositions are not attackable.

(b) The calculus $Q_{\mathrm{eff}}^{(l)}$ of the true object propositions is formulated in the metalanguage.

(c) The "modal part" is the link between the object language (b) and the formal metalanguage (a).

In a necessary metaproposition one can use the symbol $\vartriangle$ instead of $W \leq$ for the truth of $\overline{A^j}$ does not depend on W. Moreover, we use a metafalsum $\not\!\Lambda$: If one of the participants in a metadialog uses this argument he loses the dialog at once. So the rules of the "modal part" that are comparable with well-known modal calculi and important for probability theory [see Burghardt (1979) and Mittelstaedt 1979b)] can be written

$$\vartriangle \not\!\Lambda \leqq \not\!\Lambda$$

$$\vartriangle(A \to B) \leqq \vartriangle A \Rightarrow \vartriangle B$$

$$\vartriangle A \barwedge \vartriangle B \leqq \vartriangle(A \wedge B)$$

2.3. Proof Process and Real Semantics for Relative Modalities. Let $\mathscr{H}(S)$ or $\mathscr{H}(S^i)$ resp. be the Hilbert space of a quantum mechanical system S or S^i resp. and γ the set of all S^i.

(2.2) *Definition.* $\mathscr{M}_{\mathscr{H}(S)} := \{S^i \in \gamma | \mathscr{H}(S^i) = \mathscr{H}(S)\}$. If the system S has the preparation W we write $S = S(W)$ or $W = W(S)$; a *preparation* is thought of as a knowledge about S got by measurement, proof, and dialog processes.

(2.3) *Definition.* $\mathscr{M}_{\mathscr{H}(S)}^{(W)} := \{S^i \in \mathscr{M}_{\mathscr{H}(S)} | S^i = S^i(W)\}$. So $\mathscr{M}_{\mathscr{H}(S)}^{(W)}$ is the set of all systems S^i whose Hilbert space is the same as that of S and which have the same preparation W. Let A be a proposition concerning the system $S(W)$.

(2.4) *Definition.* A is called *true in* $S(W)$ iff $S(W) \underset{D_m}{\vdash} A$ (i.e., iff the proponent has a strategy of success for A in a material dialog if the system relevant for measurements has the preparation W).

(2.5) *Definition.* A is called *necessary in* $S(W)$ iff $W \underset{D_m}{\vdash} A$ (i.e., iff the proponent has a strategy of success for A in a material dialog if the preparation W can be used as a true premise, but the success in the dialog does not depend on the contingent properties of S).

So the relative necessity does not depend on the special system S but only on the preparation W; this leads to the following:

(2.6) *External semantics for relative necessity.* A is necessary in $S(W) \underset{\text{Int.}}{\leftrightarrow} \forall S^i \in \mathcal{M}^{(W)}_{\mathcal{H}(S)}$: $S^i(W) \underset{D_m}{\vdash} A$

By $\underset{\text{Int.}}{\leftrightarrow}$ I wish to emphasize that (2.6) is only an interpretation—Mittelstaedt (1979b) calls it an "ensemble theoretic interpretation"—in contrast to the definition (2.5). Of course this interpretation depends on the size of the set that is presupposed by Mittelstaedt (1979b) as "sufficiently large finite or infinite."

If one defines an equivalence relation $\simeq$ on the set $\mathcal{M}_{\mathcal{H}(S)}$ by

$$S^i \simeq S^j \cap W(S^i) = W(S^j)$$

the above interpretation for the relative necessity can be written

(2.7) A is *necessary in* $S(W) \underset{\text{Int.}}{\leftrightarrow} \forall S^i \in \gamma$: $(S^i \simeq S \cap S^i \underset{D_m}{\vdash} A)$ and in an analogous way for the relative possibility

(2.8) A is *possible in* $S(W) \underset{\text{Int.}}{\leftrightarrow} \exists S^i \in \gamma$: $(S^i \simeq S$ and $S^i \underset{D_m}{\vdash} A)$

A "possible-world interpretation" is proposed in Mittelstaedt (1980) in which W is presupposed as a *maximal* knowledge, i.e., a pure state. In contrast to the knowledge about a classical system W can never be *perfect* and therefore there are propositions objectively undecidable.

W is extended by parameters $\lambda^{(\nu)}$ such that "possible worlds" $(W, \lambda^{(\nu)})_S$ represent dispersion-free states of the system S in which each proposition A about S is determined, i.e.,

$$\Delta_{(W,\lambda^{(\nu)})_S} A \qquad \text{or} \qquad \Delta_{(W,\lambda^{(\nu)})_S} \neg A$$

The *real situation* of S is given, say, by $(W^w, \lambda^w)_S$.

(2.9) *Definition.* (i) $\mathcal{W}_S := \{(W_\rho, \lambda^{(\nu)})_S | \nu \in I, \rho \in J\}$ (with some index sets I and J) is the set of all "possible worlds" of S. (ii) $\mathcal{W}_S^{(W)} := \{(W_\rho, \lambda^{(\nu)})_S \in \mathcal{W}_S | W_\rho = W\}$ is the set of all "possible worlds" of S corresponding to the same state W.

(2.10) *Internal semantics for relative necessity.* A is necessary in $(W^w, \lambda^w)_S \underset{\text{Int.}}{\leftrightarrow}$

$$\forall \left(W_\rho, \lambda^{(\nu)}\right)_S \in \mathcal{W}_s\colon (W^w, \lambda^{(\nu)})_S \underset{D_m}{\vdash} A$$

i.e., A is necessary in the "real situation" $(W^w, \lambda^w)_S$ of S iff in each "possible world" $(W^{w}, \lambda^{(\nu)})_S$—corresponding to the same state W^{w} as the "real situation"—the proponent has a strategy of success in the material dialog about A.

Using the equivalence relation $\approx$ defined on $\mathscr{W}_S$ by

$$\left(W_\rho, \lambda^{(\nu)}\right)_S \approx \left(W_\sigma, \lambda^{(\mu)}\right)_S \cap W_\rho = W_\sigma$$

the external semantics for the relative necessity and—in an analogous way—for the relative possibility can be represented by the following:

(2.11) A is *necessary in* $(W^{w}, \lambda^{w})_S \underset{\text{Int.}}{\leftrightarrow}$

$$\forall\left(W_\rho, \lambda^{(\nu)}\right)_S \in \mathscr{W}_S: \left(\left(W_\rho, \lambda^{(\nu)}\right)_S \approx (W^{w}, \lambda^{w})_S \right.$$

$$\left. \cap \left(W_\rho, \lambda^{(\nu)}\right)_S \underset{D_m}{\vdash} A\right)$$

(2.12) A is *possible in* $(W^{w}, \lambda^{w})_S \underset{\text{Int.}}{\leftrightarrow}$

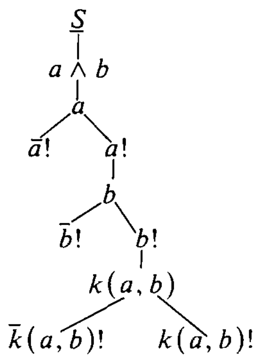
$$\exists\left(W_\rho, \lambda^{(\nu)}\right)_S \in \mathscr{W}_S: \left(\left(W_\rho, \lambda^{(\nu)}\right)_S \approx (W^{w}, \lambda^{w})_S \quad \text{and} \quad \left(W_\rho, \lambda^{(\nu)}\right)_S \underset{D_m}{\vdash} A\right)$$

Comparing (2.11) with (2.7) and (2.12) with (2.8) it can be seen that instead of the system S^i in the external semantics we have the parameters $\lambda^{(\nu)}$ in the internal semantics.

It must be emphasized that $\lambda(\nu)$ *cannot* attach objectively a value of an observable to the system not yet given by W; this would lead to a contradiction to the well-known non-hidden-variable theorems. The parameter does not concern the values in the *real* situation of the system but rather the predictions of possible measurement outcomes.

The above interpretations are also called *proof process semantics* for modalities because we used the material dialog that is the frame for the proof of a proposition. Stachow (1981) proposes a *real semantics* equivalent to the proof process semantics.

A proof process is a branch in a game tree which represents all possible proof processes. For example, consider the proposition $a \wedge b$ concerning the system S:



The proof of $a \wedge b$ fails of course by a disproof of a, i.e., $\bar{a}!$, for instance, and the only successful proof is $\langle a!, b!, k(a, b)! \rangle$. The proof of each elementary proposition is a yes–no experiment that gives us the information if the system S has the corresponding property E_a or not.

(2.13) *Definition.* (a) A *material process* is an n-Tupel $\langle E_1, \ldots, E_n \rangle$ of properties of a system S represented in the game tree by a branch[1] $P := \{ p \}$ is the set of all material processes and $S \vdash p$ is an abbreviation for the statement "A material process relative to the system S is performed."

(b) $P(A)$ is the set of all material processes corresponding to *successful* branches in the game tree. Example: $P(a \wedge b) = \{ \langle E_a, E_b, E_{k(a,b)} \rangle \}$.

(c) A proposition A is *true relative to* S iff $S \vdash p$ and $p \in P(A)$.

(d) The *physical state* p_0 is the material process that describes the whole "history," i.e., the preparation, of the system. In this sense the knowledge given by p_0 is said to be *maximal*; it can represent, e.g., a pure state or a mixture. (This must be distinguished from Mittelstaedt's terminology always using "maximal" for the knowledge representing a *pure* state.) $P_0 := \{ p_0 \}$ is the set of all pure physical states.

(e) The operation $\uplus$ is defined by $\uplus : P' \to P_0$ with

$$P' = \{ (p_0, p) \in P_0 \times P \mid p_0 = \langle E_a, \ldots, E_b \rangle \in P_0 \text{ and } p = \langle E_c, \ldots, E_d \rangle \in P \cap \langle E_a, \ldots, E_b, E_c, \ldots, E_d \rangle \in P_0 \}$$

and $(p_0, p) = (\langle E_i \rangle, \langle E_j \rangle) \to \langle E_i, E_j \rangle$

(f) For each proposition A the *successor relation* N_A is defined on the set P_0 by

$$q_0 N_A p_0 \quad \text{iff} \quad \exists p \in P(A) \cup P(\neg A) \colon q_0 = p_0 \uplus p$$

[1] This branch corresponds to a physically realizable process.

(2.14) *Real semantics for relative necessity.* A is necessary relative to p_0 iff

$$\forall p' \in P: (p_0 \Cup p' N_A p_0 \cap p' \in P(A))$$

(I.e., iff for each successor $p_0 \Cup p'$ of p_0 the material process p' corresponds to a successful branch in the game tree.)

Here we have presupposed that p_0 represents a maximal knowledge about the system; in the more general case Stachow uses a knowledge $W = P(A')$ instead of p_0 which includes a *subjective ignorance*: A' is a proposition about the system representing a nonmaximal knowledge, i.e., A' does not contain the whole "history" of the system. In the set $P(A')$ of material processes corresponding to successful branches in the game tree of A' there are physical states p_0 that are the starting points for a proof of the relative necessity of a proposition A:

(2.15) A is necessary relative to $W = P(A')$ iff

$$\forall p_0 \in W \, \forall p' \in P: (p_0 \Cup p' N_A p_0 \cap p' \in P(A))$$

A is possible relative to $W = P(A')$ iff

$$\exists p_0 \in W \exists p' \in P: (p_0 \Cup p' N_A p_0 \text{ and } p' \in P(A))$$

Remark. I think the expression "real semantics" is not correct because only *one* branch of the game tree can be a part of the reality. The name "potentiality semantics" would be more adequate for the material processes $p' \in P$ are *potentialities* of the reality.

2.4. The Representation of Modalities in the Hilbert Space. The propositions A about a quantum mechanical system S are represented in the appropriate Hilbert space by subspaces M_A or projection operators P_A on M_A, respectively, and an implication $A \to B$ is materially or formally true iff $M_A \subseteq M_B$ or $P_A \leqslant P_B$. Therefore we have for the relative *necessity*

$$\triangle_W A \cap P_W \leqslant P_A \cap P_W = P_W P_A \cap P_W = P_A P_W P_A$$

and

$$\triangle_W A \cap M_W \subseteq M_A \cap \forall \varphi \in \mathscr{H}: (\varphi \in M_W \cap \varphi \in M_A)$$

The *necessary negation* is represented by

$$\vartriangle_W \neg A \cap M_W \subseteq M_{\neg A} \cap M_W \perp M_A$$

$$\cap \forall \psi \in M_W \forall \varphi \in M_A :\ \psi \perp \varphi$$

$$\cap \forall \varphi \in M_A (\psi \perp \varphi \forall \psi \in M_W)$$

$$\cap \forall \varphi \in \mathscr{H} (\varphi \in M_A \cap \varphi \perp M_W)$$

The relative *possibility* is defined by

$$\triangledown_W A : \cap \daleth \vartriangle_W \neg A$$

and we have

$$\triangledown_W A \cap \daleth P_W = P_W P_{\neg A} \cap \daleth P_W = P_W (1 - P_A) \cap P_W P_A \neq 0$$

and

$$\triangledown_W A \cap \daleth (\forall \varphi \in \mathscr{H} (\varphi \in M_A \cap \varphi \perp M_W))$$

$$\cap \exists \varphi \in \mathscr{H} \left(\varphi \in M_A \text{ and } \varphi \not\perp M_W \right)$$

These results will be compared in Section 4 with Dalla Chiara's modalities.

Further modalities and their Hilbert space representation can be found in Mittelstaedt (1979b).

3. MODAL INTERPRETATION OF QUANTUM MECHANICS (VAN FRAASSEN)

3.1. Why and to What End a Modal Interpretation? As is well known there are some difficulties in the discussion of the quantum mechanical measuring process [cf. Mittelstaedt (1976)]. Two arguments used in such a consideration are the *projection postulate*,

PP: "If an O measurement is made on some system in state $\theta = \Sigma d_i |0_i\rangle$, then the system undergoes a transition to a state $|0_k\rangle$ with probability d_k^2."

and the *ignorance postulate*,

IP: "If a system is in mixed state ρ, then it is really in one of the proper eigenstates of ρ."

Van Fraassen (1972) argued that both PP and IP together with some other postulates lead to inconsistencies wherefore he refused PP and IP: "I now accept the postulates called composition, evolution, reduction and Born, with no restrictions on their scope; reject projection and ignorance; and develop an interpretation according to which the phenomena are as *if* projection and ignorance were true [cf. van Fraassen (1976)]."

He introduced a formal language the syntax of which includes a symbol $\Box$ for "necessary" and called it a "modal interpretation of quantum mechanics." In particular, he gave an interpretation of the EPR paradox by this modal language. Recently van Fraassen (1979) deepens his modal concept giving an axiomatic approach to his modal interpretation.

3.2. The "Copenhagen Modal Interpretation" (CMI). *3.2.1. Mixtures.* We consider a state ρ describing the *dynamical* state of the system in question by which predictions about the evolution of this system can be made according to the laws of quantum mechanics. If ρ is a pure state ($\rho = \rho^2$) the information about the system given by ρ is complete. (In the terminology of Mittelstaedt the knowledge given by ρ is *maximal* and cannot be refined up to a *perfect* knowledge.)

If ρ is not pure, i.e., ρ is a mixture, there are decompositions $\rho = \Sigma c_i P_{\phi_i}$ with pure states ϕ_i and projections P_{ϕ_i} onto the one-dimensional subspaces $[\phi_i]$. All such pure states ϕ_i are comprised in the set U_ρ. As mentioned above van Fraassen rejects the ignorance interpretation and premises the concept of a mixture "as being equally basic and objective as pure states." Furthermore he prefers the so-called

(3.1) *Copenhagen interpretation* of a mixture. "If the dynamical state ρ at time t is pure it provides complete information about the values of the observables at that time; but if the state is mixed, the information is incomplete."

In order to complete the information in the case of a mixture van Fraassen uses a parameter λ that is a pure state in U_ρ. The *model* (ρ, λ) describes the *actual* situation of the system in the mixed state ρ. Of course this concept can also be used in the case of a pure state $\rho = \rho^2$ when λ coincides with ρ and the model (ρ, λ) does not give more information then the dynamical state ρ itself.

This parameter λ refines the information of ρ partly up to the information of a pure state: λ can only be used to describe the *actual* situation but not to make predictions about the evolution of the system. The latter can only be done by the mixture ρ. So, van Fraassen introduces two different kinds of propositions about a physical system:

(3.2) Let m be an observable, E a Borel set on the real line ($E \in \mathscr{B}(\mathbb{R})$) and

$$P_m^\alpha(E) = \sum_{\Lambda \in E} \mathrm{Tr}(\alpha \mathscr{P}_{\mathscr{U}_\Lambda})$$

the Born probability to find a value of m in E if the system is in state α; Λ is an eigenvalue of m and $\mathscr{P}_{\mathscr{U}_\Lambda}$ the projection on the corresponding eigenspace. (i) The *state attribution* $[m, E]$ ("m must have a value in E") is true in $\omega = (\rho, \lambda)$ iff $P_m^\rho(E) = 1$. (ii) The *value attribution* $\langle m, E \rangle$ ("m actually has a value in E") is true in $\omega = (\rho, \lambda)$ iff $P_m^\lambda(E) = 1$. This is called by van Fraassen the *Copenhagen postulate*.

In the case of a pure state $\rho = \rho^2 = \lambda$ a value attribution $\langle m, E \rangle$ is true in ω iff the corresponding state attribution $[m, E]$ is true in ω, i.e., there is no further information by λ in the case of a pure state in agreement with the "Copenhagen interpretation." But if ρ is a mixture we only have the implication: If $[m, E]$ is true in ω then $\langle m, E \rangle$ is true in ω too; i.e., there are value attributions $\langle m', E' \rangle$ being true though the corresponding state attributions $[m', E']$ are not true. The truth of $\langle m', E' \rangle$ in this case is yielded by the new parameter λ.

Van Fraassen (1973) emphasizes that the parameters λ are no hidden variables in the sense of von Neumann, Kochen–Specker, Jauch–Piron, or Bohm–Bub, though there are some connections to these *hidden variable theories*.

3.2.2. Quantum Logic. In (3.2) we introduced two kinds of attributions, viz. $[m, E]$ and $\langle m, E \rangle$; now we define the corresponding *propositions*:

(3.3) (i) $|[m, E]| := \{\omega | [m, E] \text{ is true in } \omega\}$ is a *state attribution proposition*. $\mathbb{P} := \{[m, E]\}$ is the set of all state attribution propositions.

(ii) $|\langle m, E \rangle| := \{\omega | \langle m, E \rangle \text{ is true in } \omega\}$ is a *value attribution proposition*. $\mathbb{V} := \{|\langle m, E \rangle|\}$ is the set of all value attribution propositions.

Defining the operations

$$|[m, E]|^\perp = |[m, \mathbb{R} - E]|$$

$$|[m, E]| \dot{\cup} |[m, F]| = |[m, E \cup F]|$$

$$|[m, E]| \dot{\cap} |[m, F]| = |[m, E \cap F]|$$

and the relation

$$|[m, E]| \dot{\subseteq} |[m', E']| \quad \text{iff} \quad P_m^\alpha(E) \leqslant P_{m'}^\alpha(E') \text{ for all } \alpha,$$

the algebraic structure $\langle \mathbb{P}, \dot{\subseteq}, \perp \rangle$ can be shown to be an *orthoposet* that is a "*quantum logic*" in the sense of G. Hardegree. On the set $\mathbb{V}$ another relation $\dot{\leqslant}$ can be defined and $\langle \mathbb{V}, \dot{\leqslant}, \perp \rangle$ is an orthoposet too isomorphic to $\langle \mathbb{P}, \dot{\subseteq}, \perp \rangle$.

3.2.3. Modalities. An *accessibility relation R* on the set of all states is defined by the following:

(3.4) $\alpha R \beta$ iff α is a mixture of β and another state γ: $\alpha = c\beta + (1-c)\gamma$ with $c \in (0, 1]$.

Ochs (1979) calls β a *convex component* of α iff $\alpha R \beta$. It can be shown that this is equivalent to the relation $\tilde{R}$ for models:

(3.5) $\omega \tilde{R} \omega'$ iff $(\omega \in q \cap \omega' \in q)$ for all $q \in \mathbb{P}$. We call ω' *possible relative to* ω.

Now we are able to introduce the modal operators $\Box$, $\Diamond$, $\boxdot$, and $\diamonddot$; for any $q \in \mathbb{P}V$ we define the following:

$$\begin{aligned}
(3.6) \qquad & \Box q = \{\omega | (\omega \tilde{R} \omega' \cap \omega' \in q) \text{ for all } \omega'\} \\
& \Diamond q = \{\omega | (\omega R \omega' \cap \omega' \in q) \text{ for some } \omega'\} \\
& \boxdot q = \{\omega = (\alpha, \lambda) | (\alpha = \beta \cap \omega' \in q) \text{ for all } \omega' = (\beta, \mu)\} \\
& \diamonddot q = \{\omega = (\alpha, \lambda) | (\alpha = \beta \cap \omega' \in q) \text{ for some } \omega' = (\beta, \mu)\}
\end{aligned}$$

and q is called *necessary* (*possible*, *dot necessary*, or *dot possible*, resp.) iff $\Box q = q$ ($\Diamond q = q$, $\boxdot q = q$ or $\diamonddot q = q$, resp.)

So we have that

$|[m, E]|$ is necessary

$$\cap \{(\alpha, \lambda) | (\alpha R \beta \cap (\beta, \mu) \in |[m, E]|) \forall (\beta, \mu)\} = |[m, E]|$$

$$\cap \forall \alpha \text{ with } P_m^\alpha(E) = 1 \; \forall \beta \colon \alpha R \beta \cap (\beta, \mu) \in |[m, E]|$$

This suggests a definition of a *relative* necessity:

(3.7) *Definition.* (i) $|[m, E]|$ is *necessary relative to* the model $\omega = (\alpha, \lambda)$ in which $[m, E]$ is true, i.e., $P_m^\alpha(E) = 1$

$\cap (\alpha R \beta \cap (\beta, \mu) \in |[m, E]|) \qquad$ for all $\omega = (\beta, \mu)$

$\cap$ If α is a mixture of β and some other state γ, we have $P_m^\beta(E) = 1$.

Of course $|[m, E]|$ is necessary relative to each model (α, λ) with a pure state $\alpha = \alpha^2 = \lambda$ in which $[m, E]$ is true; i.e., with respect to a pure state there is no difference between truth and relative necessity.

(ii) $|\langle m, E\rangle|$ is *necessary relative to* the model $\omega = (\alpha, \lambda)$ in which $\langle m, E\rangle$ is true, i.e., $P_m^\lambda(E) = 1$

$\curvearrowleft$ If α is a mixture of β and some other state γ and if $\mu \in U_\beta$, we have $P_m^\mu(E) = 1$.

(iii) $|\langle m, E\rangle|$ is *dot necessary relative to* the model (α, λ) in which $\langle m, E\rangle$ is true

$$\curvearrowleft P_m^\mu(E) = 1 \qquad \text{for all } \mu \in U_\alpha$$

Nevertheless, the three *definientes* in (i), (ii), and (iii), resp., are equivalent, and so the relative necessities and the relative dot necessity cannot be distinguished semantically. This result leads to a connection of the two orthoposets $\mathbb{P}$ and $\mathbb{V}$ by the modal operators:

$$\Box|[m, E]| = \boxdot|[m, E]| = \Box|\langle m, E\rangle| = \boxdot|\langle m, E\rangle| \tag{3.8}$$

Furthermore we know that $|[m, E]|$ is necessary:

$$|[m, E]| = \Box|[m, E]| \tag{3.9}$$

Equations (3.8) and (3.9) together can be interpreted as follows:

(3.10) A state attribution is true in the model (α, λ) (i) iff it is necessary relative to this model, (ii) iff the corresponding value attribution proposition is necessary relative to this model.

On the other hand value attribution propositions generally are not necessary.

3.3. A Comparison of van Fraassen's Necessity with Mittelstaedt's Modalities. Considering the concept of relative necessity within the approaches of Mittelstaedt and van Fraassen a serious difference can be stated in the case of a system in a *pure* state. According to van Fraassen it does not make any sense to say that a proposition A about this system is necessary except that A is true. On the contrary Mittelstaedt emphasizes that it is just the pure state (with a maximal knowledge about the system) which is the presupposition of a meaningful concept for the modalities.

This fundamental difference is reflected by the hidden variable interpretations of the necessity: As seen in Section 2.3. Mittelstaedt's internal semantics for relative necessity starts with a pure state and uses the parameters $\lambda^{(\nu)}$ in order to cover all possible outcomes of a measurement. On the other hand, van Fraassen's parameters λ are the pure states in a decomposition of a mixed state. Nevertheless, we introduce a *necessity operator* $\tilde{\Delta}$ and a *possibility operator* $\tilde{\nabla}$ within the approach of Mittelstaedt analogous to the way van Fraassen does; $\tilde{\Delta}$ and $\tilde{\nabla}$ resp. are defined on the set of subsets S of the given Hilbert space $\mathscr{H}$ in the following way:

$$(3.11) \qquad \tilde{\Delta}S := \{\varphi \in \mathscr{H} \mid \varphi \in S\}$$

$$\tilde{\nabla}S := \left\{\varphi \in \mathscr{H} \mid \exists \psi \in S \colon \psi \not\perp \varphi\right\}$$

Using the results of Section 2.4 we get for a subspace $S = M_A$ corresponding to a proposition A

$$\tilde{\Delta}M_A = \{\varphi \in \mathscr{H} \mid \Delta_\varphi A\}$$

$$\tilde{\nabla}M_A = \{\varphi \in \mathscr{H} \mid \nabla_\varphi A\}$$

that motivates our definition of $\tilde{\Delta}$ and $\tilde{\nabla}$. Of course (3.11) is equivalent to

$$(3.12) \qquad \tilde{\Delta}S = S$$

$$\tilde{\nabla}S = \mathscr{H} - S^{\perp} \qquad (\tilde{\nabla}M_A = \mathscr{H} - M_{\neg A})$$

The state attribution proposition $\|[m, E]\|$ can be identified with the subspace M_A corresponding to $A = [m, E]$ and we get

$$(3.13) \qquad \tilde{\Delta}\|[m,E]\| = \|[m,E]\| = \Box\|[m,E]\|$$

So, at least in a certain formal sense, the necessity operators of Mittelstaedt and van Fraassen are equivalent.

4. MODAL TRANSLATION OF QUANTUM LOGIC (DALLA CHIARA)

4.1. The Embedding of Intuitionistic into Modal Logic. In 1932 the intuitionistic logic was embedded into the well-known modal system $S4$ by K. Gödel (1932) and it became possible to prove the validity of a proposition of a nonclassical system within a "classical" modal system. $S4$ is "classical" insofar as it is a modal extension of the classical propositional calculus. The embedding can be performed by different functions, e.g., by

Gödel's interpretation μ that is a function on the set $\mathscr{L}$ of well-formed formulas (wffs) in $\neg$, $\vee$, $\wedge$, $\rightarrow$ into the set $\mathscr{L}_\wedge$ of modal wffs in $\sim$, &, $\vee$, $\supset$, L [see Rautenberg (1979) and Figure 1] with

$$\mu(P)=\begin{cases} Lp & \text{if } P=p \text{ is a variable}\\ L\sim\mu(Q) & \text{if } P=\neg Q\\ \mu(Q)\&\,\mu(R) & \text{if } P=Q\wedge R\\ \mu(Q)\vee\mu(R) & \text{if } P=Q\vee R\\ L(\mu(Q)\supset\mu(R)) & \text{if } P=Q\rightarrow R\end{cases}$$

One can prove that a proposition P is valid in the intuitionistic logic iff its "modal translation" $\mu(P)$ is valid in S4:

$$\overset{i}{\models} P \quad \text{iff} \quad \overset{S4}{\models} \mu(P)$$

Of course this is a remarkable mathematical result. But this embedding can neither help us "to understand better the sense of intuitionistic propositions" as pretended by Rautenberg (1979)—for the usage of such vague notions as *necessary* and *possible* cannot enrich our understanding—nor is the nonclassical property of the intuitionistic logic unimportant; indeed $S4$ is an extension of the classical logic but the modalities are totally new compared to classical logic.

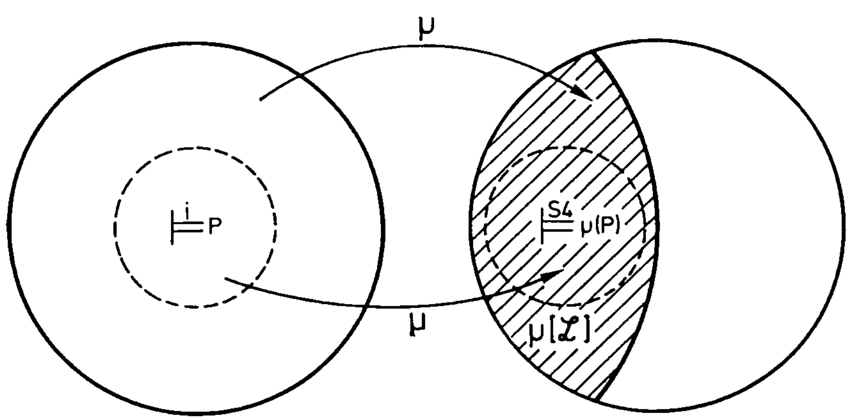


Fig. 1. Embedding of the intuitionistic logic i into the model logic $S4$. The big circles represent the sets of nonmodal wffs (on the left) and of modal wffs (on the right) resp.

4.2. The Embedding of Quantum Logic into Modal Logic. An embedding similar to the one mentioned above was done by Dalla Chiara (1977) with her quantum logic. She built up a language about a quantum mechanical system which we will consider later and translated its logic into a modal language in order to show that "from a logical point of view quantum logic is not really essential to the logic development of quantum mechanics, since the role played by this logic can be equivalently substituted by a form of classical modal logic." In a more recent paper Dalla Chiara (1979) again considered modal logic in order to "explain" special features of quantum logic: She wants "to analyse, in the particular case of quantum logic, a possible explanation of the failure of the Lindenbaum-property, by referring to a modal interpretation of this logic." However, concerning the embedding problem Dalla Chiara (1977) writes: "...can we really assert that we have completely reduced quantum logic to classical logic?... B^B and $B^B + \rho(\text{Orth})$ are not simply classical logic. Indeed they represent two very particular extensions of classical logic." That is exactly what I said above: A "classical" modal calculus is totally different from a classical logic.

Extending the work of Goldblatt (1974), who only considered propositional logic without quantifiers, Dalla Chiara begins with a first-order language $\mathscr{L}$ about a quantum mechanical system whose *atomic sentences* $P_i\mathbf{b}$ are "The value of the observable Q_i (corresponding to the one-placed predicate P_i) is in the Borel set b." The semantics of this language is induced by the complete orthocomplemented quasimodular lattice $V(\mathscr{H})$ of subspaces of the Hilbert space $\mathscr{H}(S)$. This lattice is a part of the *algebraic quantum orthorealization*

$$\mathfrak{a} = \langle V(\mathscr{H}), v, \mathscr{B}(\mathbb{R})\rangle$$

$\mathscr{B}(\mathbb{R})$ is the set of all Borel sets on the real line and v is a *valuation* function from the sentences of $\mathscr{L}$ into $V(\mathscr{H})$, e.g.,

$$v(P_i\mathbf{b}) = \left\{\psi \in \mathscr{H} | P_{Q_i}^{\psi}(b) = 1\right\}$$

at which $P_{Q_i}^{\psi}(b)$ is the well-known Born probability to find a value of the observable Q_i in the Borel set b if the system is in the state ψ. A sentence α of $\mathscr{L}$ is called *true in* $\mathfrak{a}$ iff $v(\alpha) = \mathscr{H}$ and *algebraically quantum valid* iff $v(\alpha) = \mathscr{H}$ for any algebraic quantum orthorealization.

In a second step Dalla Chiara introduced for each algebraic quantum orthorealization $\mathfrak{a}$ a possible-world semantics for $\mathscr{L}$ equivalent to the algebraic semantics. The "possible worlds" are the nonempty subspaces of $\mathscr{H}$, i.e., the set $I := V(\mathscr{H}) - \{0\}$ that is a part of the *Kripkian quantum* *orthorealization*

$$\mathcal{K}^{\mathfrak{a}} = \langle \mathcal{B}(\mathbb{R}), I, R, \rhd \rangle$$

R is the *nonorthogonality* relation on $V(\mathcal{H})$ and $\rhd$ is a subset of $I \times \{\alpha | \alpha$ sentence of $\mathcal{L}\}$ that determines if a sentence α is true in the world X:

$X \rhd P_i\mathbf{b}$	iff	$X \subseteq v(P_i\mathbf{b})$
$X \rhd \neg\beta$	iff	(not $Y \rhd \beta)\forall Y$ with XRY
$X \rhd \beta \wedge \gamma$	iff	$X \rhd \beta$ and $X \rhd \gamma$
$X \rhd \beta \to \gamma$	iff	$(Y \rhd \beta \cap Y \rhd \gamma)\forall Y \in I$
$X \rhd (x)\beta$	iff	$X \rhd \beta(\mathbf{b})\forall b \in \mathcal{B}(\mathbb{R})$

A sentence α of $\mathcal{L}$ is called *true in* $\mathcal{K}^{\mathfrak{a}}$ iff $X \rhd \alpha$ for all $X \in I$. Dalla Chiara has proved the equivalence

$$\alpha \text{ is true in } \mathfrak{a} \text{ iff } \alpha \text{ is true in } \mathcal{K}^{\mathfrak{a}}$$

Also a concept of *Kripke quantum validity* can be introduced and we have

α is algebraically quantum valid iff α is Kripke quantum valid.

The set of all algebraically quantum valid sentences can be covered by the

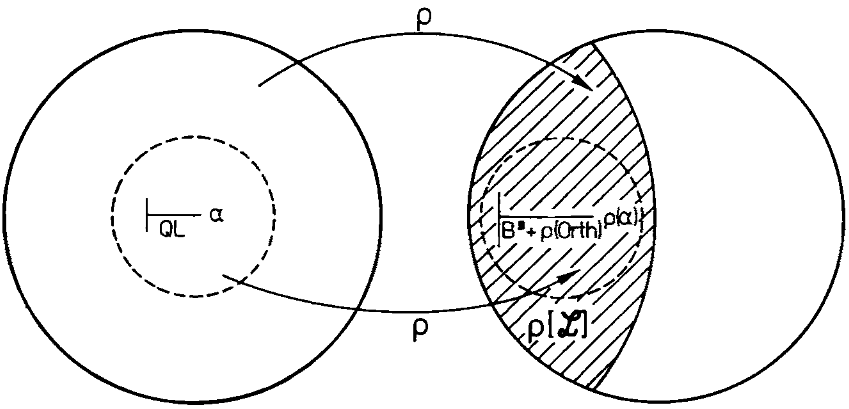


Fig. 2. Embedding of the quantum logic QL of the modal logic $B^B + \rho(\text{Orth})$ (cf. Figure 1).

calculus QL of *quantum logic.*

I wish to emphasize that within the language $\mathscr{L}$ introduced so far it is not possible to express propositions including modalities; this can be done only in the following language $\mathscr{L}_1$.

The syntax of $\mathscr{L}_1$ contains the connectives $\sim$, &, $\supset$, $\Rightarrow$, the quantifier $\wedge$ and the modal operator L. For each algebraic quantum orthorealization $\mathfrak{a}$ we introduce a semantics by a

$$B^B\text{-}Kripkian\ realization\ \mathscr{K}_1^{\mathfrak{a}} = \langle \mathscr{B}(\mathbb{R}), I, R, \vDash \rangle$$

at which R is again the nonorthogonality relation on $V(\mathscr{H})$ and $\vDash$ is a subset of $I \times \{\alpha | \alpha \text{ sentence of } \mathscr{L}_1\}$ defined by

$$\begin{array}{ll}
X \vDash P_i\mathbf{b} & \text{iff } X \subseteq v(P_i b) \\
X \vDash \sim\beta & \text{iff not } X \vDash \beta \\
X \vDash \beta \& \gamma & \text{iff } X \vDash \beta \text{ and } X \vDash \gamma \\
X \vDash \beta \supset \gamma & \text{iff } (X \vDash \beta \cap X \vDash \gamma) \\
X \vDash \beta \Rightarrow \gamma & \text{iff } (Y \vDash \beta \cap Y \vDash \gamma) \forall Y \in I \\
X \vDash \wedge x\beta & \text{iff } X \vDash \beta(\mathbf{b}) \forall b \in \mathscr{B}(\mathbb{R}) \\
X \vDash L\beta & \text{iff } Y \vDash \beta \forall Y \in I \text{ with } YRX
\end{array}$$

A sentence α of $\mathscr{L}_1$ is *true in* $\mathscr{K}_1^{\mathfrak{a}}$ iff $Y \vDash \alpha$ for any $Y \in I$ and B^B-*valid* iff α is true in $\mathscr{K}_1^{\mathfrak{a}}$ for any B^B-Kripkian realization. The set of all B^B-valid sentences can be covered by a modal calculus B^B that we will consider later.

But first we embed the quantum logic QL into the modal language $\mathscr{L}_1$ by a *translation function* ρ as follows (see Figure 2; $M := \sim L \sim$ is the operator for the *possibility*):

$$\rho(\alpha) = \begin{cases} LM\alpha & \text{if } \alpha \text{ is an atomic sentence} \\ L \sim \rho(\beta) & \text{if } \alpha = \neg\beta \\ \rho(\beta) \& \rho(\gamma) & \text{if } \alpha = \beta \wedge \gamma \\ \rho(\beta) \dashv \rho(\gamma) & \text{if } \alpha = \beta \rightarrow \gamma \\ \wedge x \rho(\beta) & \text{if } \alpha = (x)\beta \end{cases}$$

As in Section 4.1, we have the following theorem:

A sentence α of $\mathscr{L}$ is algebraically quantum valid, i.e., is deducible in the quantum logic QL, iff $\rho(\alpha)$ is deducible in the

(4.1) *modal calculus* $B^B + \rho(Orth)$ comprising [cf. Hughes and Cresswell (1968) and Dalla Chiara (1977)]

(a) the classical first-order logic;

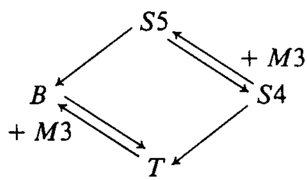
(b) the modal formation rule "If α is a wff then $L\alpha$ is a wff too"
(c) the modal transformation rule "If α is a sentence then $L\alpha$ is a sentence too";
(d) the modal axioms

$$
\begin{array}{ll}
M1: & L\alpha \supset \alpha \\
M2: & L(\alpha \supset \beta) \supset (L\alpha \supset L\beta) \\
M3: & \alpha \supset LM\alpha \\
\rho(\text{Orth}): & (\alpha \& \sim \beta) \supset M(\alpha \& L \sim (\alpha \& \beta))
\end{array}
$$

$M1$ and $M2$ together with the classical propositional calculus form the modal system T; adding $M3$ we get the *Brouwerian system* B. If we use the classical first-order language as the nonmodal basis the

(4.2) *Barcan formula* $\wedge xL\alpha \supset L \wedge x\alpha$

is deducible in the corresponding Brouwerian system. Neither B is contained in the well-known calculus $S4$ nor vice versa, but B is contained in $S5$ that is equivalent to $S4 + M3$:



Concerning the semantics there are two differences between the Kripkian semantics of Dalla Chiara and the Kripke semantics for T, $S4$, $S5$:

(1) The *accessibility relation* R is reflexive and symmetrical but not transitive. (It is reflexive in T, reflexive and transitive in $S4$, reflexive, symmetrical, and transitive in $S5$)

(2) Dalla Chiara uses a complete orthomodular lattice for the "possible worlds" instead of a set without any algebraic structure.

A completeness and soundness proof with respect to the Kripkian semantics was given by Goldblatt (1974) for the propositional calculus B. In the predicate language with quantifiers such a proof is a little bit easier if the Barcan formula is valid. This should be the reason why Dalla Chiara emphasizes the validity of the Barcan formula and uses the notation B^B, i.e., *Brouwer – Barcan*.

4.3. A Comparison of Dalla Chiara's Modalities with Those of Mittelstaedt and van Fraassen. Let β be a proposition about a physical system. The proposition "β is *necessary*" cannot be expressed in Dalla

Chiara's nonmodal object language $\mathscr{L}$. Therefore β has to be translated into the modal language $\mathscr{L}_1$ and we have the following:

(4.3) $\rho(\beta)$ is necessary in the world X

$$\cap X \models L\rho(\beta)$$

$$\cap \forall YRX\colon Y \models \rho(\beta)$$

$$\cap \forall Y \not\perp X\colon Y \subseteq v(\beta)$$

$$\cap \forall Y \in V(\mathscr{H})\colon \left(Y \not\perp X \cap Y \subseteq v(\beta)\right)$$

$$\cap \forall \varphi \in \mathscr{H}\colon \left(\varphi \not\perp X \cap \varphi \in v(\beta)\right)$$

In the language of Mittelstaedt we had

The proposition A is necessary relative to W

$$\cap \forall \varphi \in \mathscr{H}\colon (\varphi \in M_W \cap \varphi \in M_A)$$

or equivalently in the notation of Dalla Chiara:

(4.4) β is necessary relative to X

$$\cap \forall \varphi \in \mathscr{H}\colon (\varphi \in X \cap \varphi \in v(\beta))$$

Comparing (4.3) and (4.4) we have

$$X \models L\rho(\beta) \cap \Delta_X \beta \tag{4.5}$$

i.e., if a proposition is necessary in the sense of Dalla Chiara it is necessary in the sense of Mittelstaedt too. The inverse of the implication in (4.5) is generally not true.

Before discussing the probability we must consider the *necessary negation* that is very important.

If Dalla Chiara wishes to express the proposition "not β" she uses the object language $\mathscr{L}$:

(4.6) "Not β" is true in the world X

$$\cap X \rhd \neg \beta$$

$$\cap \forall YRX\colon \text{not } Y \rhd \beta$$

$$\cap \forall Y \not\perp X\colon \text{not } Y \subseteq v(\beta)$$

$$\cap \forall Y \in V(\mathscr{H})\colon (Y \subseteq v(\beta) \cap Y \perp X)$$

$$\cap \forall \varphi \in \mathscr{H}\colon (\varphi \in v(\beta) \cap \varphi \perp X)$$

The same result is got by translating $\neg\beta$ in the modal language $\mathscr{L}_1$ and considering $X \models \rho(\neg\beta)$. The latter is equivalent with $X \models L \sim \rho(\beta)$ which shows that a negation of an object proposition is *necessary* if we use the modal language!

As shown in Section 2.4 the necessary negation $\vartriangle_W \neg A$ in Mittelstaedt's approach is represented in the Hilbert space by (using the notation of Dalla Chiara)

$$(4.7) \qquad \vartriangle_X \neg\beta \cap \forall\varphi \in \mathscr{H}\colon (\varphi \in v(\beta) \cap \varphi \perp X)$$

So we have the equivalence of the necessary negation

$$(4.8) \qquad X \vartriangleright \neg\beta \cap X \models L \sim \rho(\beta) \cap \vartriangle_X \neg\beta$$

At least we have to consider the *possibility* defined as the negation of the necessary negation. Once again in Dalla Chiara's approach the modal language must be used in order to express the possibility.

(4.9) β is possible in the world X

$$\cap X \models M\rho(\beta)$$

$$\cap X \models \sim L \sim \rho(\beta)$$

$$\cap \text{ not } X \models L \sim \rho(\beta)$$

$$\cap \text{ not}(\forall Y \in V(\mathscr{H})\colon (Y \subseteq v(\beta) \cap Y \perp X))$$

$$\cap \exists Y \in V(\mathscr{H})\colon \left(Y \subseteq v(\beta) \text{ and } Y \not\perp X\right)$$

The possibility of Mittelstaedt can be formulated in the notation of Dalla Chiara by

$$(4.10) \qquad \triangledown_X \beta \cap \exists \vartriangle_X \neg\beta$$

$$\cap \text{ not}(\forall\varphi \in \mathscr{H}\colon (\varphi \in v(B) \cap \varphi \perp X))$$

$$\cap \exists\varphi \in \mathscr{H}\colon \left(\varphi \in v(\beta) \text{ and } \varphi \not\perp X\right)$$

(4.9) and (4.10) yield the equivalence of the possibilities

$$(4.11) \qquad X \models M\rho(\beta) \cap \triangledown_X \beta$$

But the question arises as to why the possibilities can be equivalent though the necessities are different and possibility is defined by necessity.

To answer this question we have to investigate the semantics of necessity and negation. The negation of Dalla Chiara within the modal language is a classical "not" and the necessity is a "possible-world necessity" that refers to other worlds accessible from the given one. Mittelstaedt on the other hand uses a nonclassical negation in the object language, viz. the orthocomplement in the Hilbert space, and a "classical," i.e., non-possible-world, necessity (see Table III). The result of the first negation and the necessity is the same for both authors and is expressed in the equivalence of the necessary negation. The second negation is classical in both cases: Dalla Chiara uses again the classical negation of her modal language. Mittelstaedt—once got the metaproposition $\triangle_W \neg A$—must use the negation of his metalanguage which is a classical "not."

It is remarkable that the negation in Dalla Chiara's nonmodal object language is nonclassical too, but this negation cannot be used to express the possibility since the latter must be formulated in a modal language.

In order to compare Dalla Chiara's modalities with van Fraassen's necessity we define a *necessity operator* $\tilde{L}$ and a *possibility operator* $\tilde{M}$ on the set of subsets S of $\mathscr{H}$:

$$\text{(4.12)} \qquad \tilde{L}S := \{\psi \in \mathscr{H} | \forall \varphi \in \mathscr{H}: (\varphi \perp \psi \cap \varphi \in S)\}$$

$$\tilde{M}S := \left\{\psi \in \mathscr{H} | \exists \varphi \in S: \varphi \not\perp \psi\right\} = \mathscr{H} - S^{\perp}$$

This is motivized by the special case $S = v(\beta)$:

$$\tilde{L}v(\beta) = \{\psi \in \mathscr{H} | \psi \vDash L\rho(\beta)\}$$

$$\tilde{M}v(\beta) = \{\psi \in \mathscr{H} | \psi \vDash M\rho(\beta)\}$$

For an atomic sentence $P_i\mathbf{b}$, we can identify $v(P_i\mathbf{b})$ with $|[m, E]|$ and we get the following relation:

$$\text{(4.13)} \qquad |[m, E]| = \Box|[m, E]| = \tilde{L}\tilde{M}|[m, E]|$$

that was mentioned by van Fraassen in his lecture during the Workshop on Quantum Logic (Erice, Sicily, December 1979).

TABLE III

	Mittelstaedt	Dalla Chiara
First negation	Nonclassical	Classical
Necessity	Classical	Nonclassical
Second negation	Classical	Classical

Equation (4.13) shows that Dalla Chiara's necessity $\tilde{L}$ is stronger than van Fraassen's □, in agreement with our previous results, viz. (3.13), i.e., $\tilde{\triangle}$ is equivalent with □, and (4.5), i.e., L is stronger than △.

As can be seen from (4.3) a proposition β is necessary solely in the trivial case $v(\beta)=\mathscr{H}$ if no other restrictions are made. So I think that Dalla Chiara's necessity L (or $\tilde{L}$ resp.) is too strong and the concept of Mittelstaedt's △ or van Fraassen's □ should be preferred.

ACKNOWLEDGMENTS

First of all I wish to thank very much Prof. Dr. P. Mittelstaedt for his encouragement, criticism, and improvements. Also I am grateful to Privatdozenten Dr. Ochs and Dr. Stachow for helpful discussions and Dr. D. Holdsworth and Dr. S. Bugajski for many hints given during their visits at the University of Cologne.

I am greatly indebted to Prof. Dr. E. Beltrametti for making it possible to join in the Workshop on Quantum Logic in Erice, Sicily (December 1979) where I was able to have discussions with Prof. Dr. B. C. van Fraassen and Prof. Dr. M. L. Dalla Chiara, to whom many thanks are due.

The financial support of the Deutsche Forschungsgemeinschaft (DFG) is gratefully acknowledged.

REFERENCES

Burghardt, F. J. (1979). *Modale Quantenmetalogik mit dialogischer Begründung*. Dissertation, Cologne.

Burghardt, F. J. (1980). Modal quantum logic and its dialogic foundation, *Inter. J. Theor. Phys.*, **19**, 843–866.

Carnap, R. (1934). *Logische Syntax der Sprache*. Springer, Vienna.

Dalla Chiara, M. L. (1977). Quantum logic and physical modalities, *J. Phil. Log.*, **6**, 391–404.

Dalla Chiara, M. L. (1979). Some metalogical pathologies of quantum logic, in *Current Issues in Quantum Logic*, E. Beltrametti and B. C. van Fraassen, eds., Plenum Press, New York.

van Fraassen, B. C. (1972). A formal approach to the philosophy of science, in pp. 303–368. *Paradigms and Paradoxes*, R. Colodny, ed. University of Pittsburgh Press, Pittsburgh.

van Fraassen, B. C. (1973). Semantic analysis of quantum logic, in C. A. Hooker, ed., pp. 80–113. *Contemporary Research in the Foundations and Philosophy of Quantum Theory*. D. Reidel, Dordrecht.

van Fraassen, B. C. (1976). The Einstein–Podolski–Rosen Paradox, in *Logic and Probability in Quantum Mechanics*, P. Suppes, ed., pp. 283–301. (Synthese Library, Vol. 78) D. Reidel, Dordrecht.

van Fraassen, B. C. (1979). A modal interpretation of quantum mechanics, in *Current Issues in Quantum Logic*, E. Beltrametti and B. C. Van Fraassen, eds. Plenum Press, New York.

Goldblatt, R. I. (1974). Semantical analysis of orthologic, *J. Phil. Log.*, **3**, pp. 19–35.

Hughes, G. E., and Cresswell, M. J. (1968). *An Introduction to Modal Logic*. Methuen, London.

Lorenzen, P. (1954). Zur Begründung der Modallogik, *Archiv für mathematische Logik und Grundlagenforschung*, **2**, 15–28 (*Archiv für Philosophie*, **5**, pp. 95–108.)

Mittelstaedt, P. (1976). *Philosophical Problems of Modern Physics*. Reidel, Dordrecht.

Mittelstaedt, P. (1978). *Quantum Logic*. Reidel, Dordrecht.

Mittelstaedt, P. (1979a). The Modal Logic of Quantum Logic, *J. Phil. Log.*, **8**, 479–504.

Mittelstaedt, P. (1979b). The dialogic approach to modalities in the language of quantum physics, in *Current Issues in Quantum Logic*. E. Beltrametti and B. C. van Fraassen, eds. Plenum Press, New York.

Mittelstaedt, P. (1980). Wahrheit, Wirklichkeit und Logik in der Sprache der Physik, to be published.

Rautenberg, W. (1979). *Klassische und nichtklassische Aussagenlogik*. F. Vieweg & Sohn, Braunschweig-Wiesbaden.

Stachow, E.-W. (1981). Der quantenmechanische Wahrscheinlichkeitsbegriff, in J. Nitsch, J. Pfarr, and E.-W. Stachow, eds., *Grundlagenprobleme der modernen Physik — Festschrift für Peter Mittelstaedt zum 50.Geburtstag*. Bibliographisches Institut, Mannheim.

Ochs, W. (1979). *Some comments on the concept of State in Quantum Mechanics*. unpublished.

Gödel, K. (1932). "Eine Interpretation des intuitionistischen Aussagenkalküls," *Ergebnisse eines mathematischen Kolloquiums*, **4**, 39–40.